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A Further Link(s) with Inequivalent Representations in Quantum Field Theory

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Abstract

In recent years it has become more apparent how useful inequivalent representations of the CCR and CAR in quantum field theory may be in describing and explaining physical phenomena, and several properties and concepts have been stated, referred to, and/or developed in the literature on these ideas. In this paper, some of these are reviewed, and some further properties and concepts are developed as further links in understanding these inequivalent representations in quantum field theory. One of these is a statement as to what actually breaks down in some field theories in the transformation between representations which are unitarily inequivalent. This is developed using the language and ideas of point quantum mechanical invariance, since this should be more familiar to a much larger number of physicists. Also, a statement on state expectation values is developed which can be used as a criterion for the occurrence of inequivalent representations of the CCR and CAR in field theories.

1. *Introduction*

A well-known theorem of Von Neumann states that for systems with a finite number of degrees of freedom any Hilbert space specification is equivalent up to a unitary transformation to any other provided the canonical commutation (anticommutation) relations are preserved. However, for systems with infinitely many degrees of freedom (i.e., for fields) it has been shown that unitarily inequivalent representations occur. This was first shown in 1952 by L. Van Hove, but it did not draw much attention until 1955 when

Schweber, Wightman, and Gärding developed more of the mathematical details of this phenomenon. One need not be alarmed by the appearance of inequivalent representations in the theory, for it is an indispensable characteristic for explaining 'super' systems in a properly formulated theory of infinitely many degrees of freedom. Indeed, as pointed out by Hugenholtz (1969), 'Clearly, to describe all possible states of a many-particle system we cannot limit ourselves to only one representation. Infinitely many representations are necessary'. One of the recent applications is in gaining a better understanding of measurement theory (Hepp, 1972).

In this paper a further link in understanding inequivalent representations of the canonical commutation and anticommutation relations in field theory is made with respect to the typical and well-known unitary equivalence of representations in point quantum mechanics. This will come from the definitions and conditions for quantum mechanical invariance to be stated in Section 3. In the process we will also develop a useful statement which definitely links identifying inequivalent representations with state expectation values. This has been alluded to previously in the literature. In Section 2 we will basically recall what is meant by inequivalent representations in the context which we will use them and illustrate this with an example. In Section 3 we will develop the links which give further insight into inequivalent representations.

2. *Inequivalent Representations*

As is well known, using some very involved mathematics, Von Neumann (1931) showed that for systems with a finite number of degrees of freedom, i.e., for systems considered in point quantum mechanics, all irreducible representations of the commutation relations given by

$$[q_i, p_k] = i\delta_{ik}$$
$$[p_i, p_k] = [q_i, q_k] = 0 \tag{2.1}$$

($i, k = 1, 2, \ldots, N$ (N finite), that is, all representations where the operator set (q, p) forms a complete set in the Hilbert space, are equivalent up to a unitary transformation. Thus, if a set of operators (q, p) satisfies equation (2.1), and another set (q', p') satisfies $[q_i', p_k'] = i\delta_{ik}$, etc. then these sets are related by a unitary transformation U in the following manner:

$$q_k' = Uk_kU^{-1}, \qquad p_k' = Up_kU^{-1} \tag{2.2}$$

The operator algebra must, of course, be enlarged to contain the operators corresponding to intrinsic degrees of freedom such as spin.

A good illustration of this theorem is given by the following example of a change from one representation to another in ordinary quantum mechanics, and we show it to be unitary (Roman, 1965). Consider two Hilbert space descriptions of a quantum mechanical system,

H	$\bar{H}$	(Hilbert spaces)
$\|n\rangle$	$\|\bar{n}\rangle$	(states)
O	$\bar{O}$	(operators)

Due to the completeness of the basis sets we can make the expansions:

$$|n\rangle = \sum_{\bar{n}} |\bar{n}\rangle\langle\bar{n}|\, n\rangle$$

$$|\bar{n}\rangle = \sum_{n} |n\rangle\langle n|\bar{n}\rangle$$

Looking at the operators, O, and vectors, V, in $\bar{H}$ in terms of those in H, we have

$$\langle\bar{n}|O|\bar{m}\rangle = \sum_{n,\, m} \langle\bar{n}|n\rangle\langle n|O|m\rangle\langle m|\bar{m}\rangle$$

$$\langle\bar{n}|V\rangle = \sum_{n} \langle\bar{n}|n\rangle\langle n|V\rangle$$

or in shorthand notation,

$$\bar{O}_{\bar{n}\bar{m}} = T_{\bar{n}n}O_{nm}R_{m\bar{m}}$$

$$\bar{V}_n = T_{\bar{n}n}V_n$$

where

$$T_{\bar{n}n} = \langle\bar{n}|n\rangle, \qquad R_{n\bar{n}} = \langle n|\bar{n}\rangle = \tilde{T}^*_{n\bar{n}} = T^\dagger_{n\bar{n}} \tag{2.3}$$

We then have

$$T_{\bar{n}n}T_{n\bar{n}'} = \sum_{n} \langle\bar{n}|n\rangle\langle n|\bar{n}'\rangle = \langle\bar{n}|\bar{n}'\rangle = \delta_{\bar{n}\bar{n}'}$$

Consequently,

$$TT^\dagger = I, \qquad \text{or} \qquad T^\dagger = T^{-1}$$

and so T is a unitary transformation. In this case T does not represent an operator. The matrix representative of an operator is defined in a given representation, whereas T straddles the two representations. However, when the basis vectors are labelled by the same set of indices, one can define a linear operator U such that

$$|n\rangle = U|\bar{n}\rangle$$

which means that U can be represented by

$$U = \sum_{n} |n\rangle\langle\bar{n}|, \qquad U^\dagger = \sum_{n} |\bar{n}\rangle\langle n| \tag{2.4}$$

so that $UU^\dagger = I$, and U is a unitary operator (Messiah, 1966). In this case the unitary matrix $\langle\bar{n}|n\rangle$ is the matrix representing U in the H or $\bar{H}$ representation, that is,

$$\langle\bar{n}|n\rangle = \langle\bar{n}|U|\bar{n}\rangle = \langle n|U|n\rangle \tag{2.5}$$

This corresponds to mapping a Hilbert space back onto itself. Again, the transformation is unitary.

However, for systems with infinitely many degrees of freedom (i.e., for fields) not all irreducible representations of the commutation relations are unitarily equivalent as has been shown by various authors. Van Hove (1952) and Friedrichs (1953) were the first to study various representations of the canonical commutation and anticommutation relations, but the phenomenon of inequivalent representations was not given much attention by physicists until the appearance of papers by Schweber & Wightman (1955) and Haag (1955). Schweber & Wightman showed the existence of uncountably many unitarily inequivalent representations of the canonical commutation and anticommutation relations. More mathematical details were given in two papers by Gärding & Wightman (1956). Other important works on inequivalent representations for the canonical commutation and anticommutation relations have been written by Segal (1958), Araki & Woods (1963), Ezawa (1965), Araki & Wyss (1964), Klauder & McKenna (1965), and Klauder, McKenna & Woods (1966).

The main ideas of the theory of inequivalent representations can be illustrated by looking at Haag's (1961) example of a general Bose field system. Starting with a system of a finite number of degrees of freedom, say N, the Fock representation developed by Fock (1932) is used for our Hilbert space. Assume the system is specified by the $2N$ Hermitian operators, (q_k, p_k), with $k = 1, \ldots, N$, and that these satisfy the commutation relations in equation (2.1). A new set of operators, a_k, are then introduced by letting

$$a_k = (\omega_k/2)^{1/2} q_k + i(1/2\omega_k)^{1/2} p_k, \qquad (k = 1, \ldots, N) \tag{2.6}$$

where the ω_k are arbitrary, real, positive numbers. From equations (2.1) and (2.6) the commutation relations for the a's are

$$[a_i, a_k^\dagger] = \delta_{ik}, \qquad [a_i, a_k] = [a_i^\dagger, a_k^\dagger] = 0 \tag{2.7}$$

Assuming that the Fock vacuum state, $|O\rangle$, defined by

$$a_k |O\rangle = 0, \qquad \text{for all } k \tag{2.8}$$

exists, the Fock space is obtained by applying $a_k^\dagger$ to $|O\rangle$, i.e., the Hilbert space on which the a_k and $a_k^\dagger$ operate is defined as the closure of the linear space spanned by the basis vectors

$$\begin{aligned} &|O\rangle \\ &a_k^\dagger |O\rangle = |k\rangle \\ &a_k^\dagger a_j^\dagger |O\rangle = |k_j\rangle \\ &\ldots \qquad (k, j = 1, \ldots, N) \end{aligned} \tag{2.9}$$

In the case where N is finite, according to Von Neumann's theorem, the Fock representation constructed above is the only possible irreducible representation up to unitary equivalence. (We are only interested in irreducible representations of our algebra of operators since any reducible representation can be shown to decompose into a sum of irreducible ones (Von Neumann, 1940).) In other words, given the two sets of canonical operators (q_k, p_k) and (q'_k, p'_k), where

$$q'_k = q'_k(p_k, q_k), \qquad p'_k = p'_k(k_k, q_k) \tag{2.10}$$

one can always find a unitary operator U which yields

$$q'_k = Uq_kU^{-1} \qquad \text{and} \qquad p'_k = Up_kU^{-1} \qquad (k = 1, \ldots, N) \tag{2.11}$$

as indicated in equation (2.2). The Fock vacuum for the new set (p'_k, q'_k), say $|O'\rangle$, is related to the original Fock vacuum $|O\rangle$ by U,

$$|O'\rangle = U|O\rangle \tag{2.12}$$

In quantum field theory, i.e., N becomes infinite, the role of the q_k is taken by an infinite set of operators, corresponding to the values of the classical field ϕ at every point in space; in the same way, the role of the p_k is taken by the values of the conjugate field π. For a real scalar field,

$$\phi(\vec{x}, x_0 = 0) = \phi(\vec{x}) \text{ and } \pi(\vec{x}, x_0 = 0) = \pi(\vec{x}) = \left.\frac{\partial\phi(\vec{x}, x_0)}{\partial x_0}\right|_{x_0 = 0}$$

satisfying the commutation relations

$$[\phi(\vec{x}), \phi(\vec{x}')] = [\pi(\vec{x}), \pi(\vec{x}')] = 0$$
$$[\pi(\vec{x}), \phi(\vec{x}')] = -i\delta^3(\vec{x} - \vec{x}') \tag{2.13}$$

A more precise mathematical formulation of the commutation relations is obtained by smearing out our fields (ϕ, π) with an orthogonal, real set of square-integrable functions, $f_k(\vec{x})$, since the physical observables are not the values of a field at a single point, but rather averages of (ϕ, π) over certain regions of space. Thus, we introduce the operators

$$q_k = \phi(f_k) = \int d^3x\phi(\vec{x})f_k(\vec{x})$$
$$p_k = \pi(f_k) = \int d^3x\pi(\vec{x})f_k(\vec{x}) \qquad (k = 1, \ldots) \tag{2.14}$$

as well as a_k and $a_k^\dagger$ in an analogous way as in equation (2.6).

Haag then points out that one can formally construct the Fock space in the same way as formerly done for finite N. However, in the case now at hand, $N \to \infty$, we no longer have Von Neumann's theorem satisfied, and there are other inequivalent representations in which there is no state $|O\rangle$ with the property of equation (2.8). To exhibit this clearly, we follow Haag's example and transform from the original a_k to a new set of canonical operators b_k given by

$$b_k = a_k \cosh\theta_k - a_k^\dagger \sinh\theta_k$$
$$b_k^\dagger = -a_k \sinh\theta_k + a_k^\dagger \cosh\theta_k \qquad (k = 1, \ldots) \tag{2.15}$$

where for Haag's associated transformation, we take $\exp[2\theta_k] = \omega_k$. Taking a_k as in equation (2.6), we have

$$(a_k + a_k^\dagger)(1/2\omega_k)^{1/2} = q_k$$
$$-i(a_k - a_k^\dagger)(\omega_k/2)^{1/2} = p_k \qquad (k = 1, \ldots) \tag{2.16}$$

Similarly for the b's, we have

$$(b_k + b_k^\dagger)(1/2\omega_k)^{1/2} = q_k'$$
$$-i(b_k - b_k^\dagger)(\omega_k/2)^{1/2} = p_k' \qquad (k = 1, \ldots) \tag{2.17}$$

and our transformation equation (2.15) becomes in terms of $(q_k, p_k) \to (q_k', p_k')$,

$$q_k' = (1/\omega_k)^{1/2} q_k$$
$$q_k' = (\omega_k)^{1/2} p_k \qquad (k = 1, \ldots) \tag{2.18}$$

so that we retain the same commutation relations for our new operators as for the original set. By means of equation (2.15), the Fock representation of the $(a_k, a_k^\dagger)$ also defines a representation of the $(b_k, b_k^\dagger)$ in the same Hilbert space. However, it is not possible in general to carry out a Fock construction for the operators $(b_k, b_k^\dagger)$, that is, there is no vector $|O'\rangle$ in our Hilbert space which satisfies

$$b_k |O'\rangle = 0 \qquad (\text{for all } k)$$

The same is true for the unitary operator U which interlocks the operators $(b_k, b_k^\dagger)$ with the operators $(a_k, a_k^\dagger)$,

$$b_k = U a_k U^{-1} \qquad (k = 1, \ldots)$$

in spite of the fact that the b's and the a's satisfy the same commutation relations.

To explore these ideas, Haag considers the matrix elements of U in our defining Fock space. Formally, U is found to be

$$U = \exp[\tfrac{1}{2} \sum_k \theta_k (a_k^\dagger a_k^\dagger - a_k a_k)] \tag{2.19}$$

We now want to show that U is not an operator in our defining Hilbert space, for all of its matrix elements between states of this space are zero. Haag begins by considering $\langle O | U | O \rangle$. As is shown in the paper of Umezawa & Kamefuchi (1964) on 'Bose fields and inequivalent representations', $\langle O | U | O \rangle$ becomes with U given by equation (2.19),

$$\langle O | U | O \rangle = \exp[-\tfrac{1}{2} \sum_k \log \cosh \theta_k] \tag{2.20}$$

which can be written as $\pi_k (\cosh \theta_k)^{-1/2}$. Therefore, $\langle O | U | O \rangle \neq 0$ only if

$$\pi_k (\cosh \theta_k) < \infty \tag{2.21}$$

From equation (2.20) we see that if we let $N \to \infty$ and $V \to \infty$, such that $N/V = \rho$, then

$$\langle O | U | O \rangle = \lim_{\substack{N \to \infty \\ V \to \infty}} \left[\prod_k^{\infty} (\cosh \theta_k)^{-1/2} \right]$$

$$= \lim_{\substack{N \to \infty \\ V \to \infty}} [\exp \{-\tfrac{1}{2} V(1/2\pi)^3 \int d^3k \log \cosh \theta_k\}] = 0 \tag{2.22}$$

Likewise, considering $(\Psi_M, U\Psi_L)$, (where Ψ_M, Ψ_L are basis vectors of our defining space obtained from $| O \rangle$ by operating on it with the $a_k^\dagger$ M and L times, respectively), since $a_k^\dagger$, a_k commute with $a_i^\dagger$, a_i for $k \neq i$, we would have

$$(\Psi_M, U\Psi_L) = \prod_k F_k \tag{2.23}$$

where $F_k \neq (\cosh \theta_k)^{-1/2}$ at most for $(M + L)$ factors F_k. Therefore, in case equation (2.21) does not hold, the change of a finite number of factors will not change this divergence, and equation (2.23) vanishes for every finite M and L. Since the states Ψ_M form a complete set, U exists in our defining space only if equation (2.21) holds, otherwise it vanishes.

For $L = 0$, from equation (2.23), in case equation (2.21) does not hold,

$$(\Psi_M, U\Psi_0) = (\Psi_M, \Psi_0') = 0 \tag{2.24}$$

i.e., Ψ_0' does not exist in our defining space, where $\Psi_0 = | O \rangle$ and $\Psi_0' = | O' \rangle$.

Thus, this general-type example shows that the most important aspect about dynamical maps (equation (2.15) for this case) is that they are *not* generally unitarily implementable for systems with an infinite number of degrees of freedom. In particular, the very useful quantum field theories exhibiting broken symmetries are of this type, for which Umezawa's (1965) self-consistent field theory was formulated.

3. *Further Links*

In this section we develop the further links with inequivalent representations of the CCR and CAR of quantum field theory stated in the Introduction.

The test for quantum mechanical invariance given by Wightman & Barut (1959) can be summarized in the following manner. Given:

H_a	H_b	(Hilbert spaces)
$\| \Phi_a \rangle$	$\| \Phi_b \rangle$	(states)
O_a	O_b	(operators)
x^a	x^b	(coordinate relations),

then the following two statements must hold:

(1) Invariance of squares of state overlaps under the bodily transformation, B, from space 'a' to 'b', that is,

$$|\langle \Phi_a | (\Psi_\alpha)_a \rangle|^2 = |\langle B_{b\leftarrow a}\Phi_a | B_{b\leftarrow a}(\Psi_\alpha)_a \rangle|^2 \tag{3.1}$$

where $(\Psi_\alpha)_a$ are eigenstates of the observables, O_a, with quantum numbers, α, and Φ_a is any state in H_a given by $\Phi_a = \Sigma_\alpha C_\alpha | \Psi_\alpha \rangle_a$, where the C_α are expansion coefficients, and $B_{b\leftarrow a} = \langle \Phi_b | \Phi_a \rangle$. In words, the bodily identical transformation means that one can write a complete description for the same system at different orientations using space '*a*' or '*b*'.

(2) Invariance of squares of state overlaps under the subjective transformation S from '*a*' to '*b*', that is,

$$|\langle \Phi_a | (\Psi_\alpha)_a \rangle|^2 = |\langle S_{b\leftarrow a}\Phi_a | S_{b\leftarrow a}(\Psi_\alpha)_a \rangle|^2 \tag{3.2}$$

In words, the subjective transformation means that the same physical situations can be described by space '*a*' as well as space '*b*'.

Statements (3.1) and (3.2) can be combined into one statement for invariance if we consider the bodily transformation from '*a*' to '*b*' and then subjectively transform this bodily state of '*b*' back onto '*a*', so we have

$$|\langle \Phi_a | (\Psi_\alpha)_a \rangle|^2 = |\langle S_{a\leftarrow b}B_{b\leftarrow a}\Phi_a | S_{a\leftarrow b}B_{b\leftarrow a}(\Psi_\alpha)_a \rangle|^2 \tag{3.3}$$

As is well known, in quantum mechanics the transformation $U \equiv S_{a\leftarrow b}B_{b\leftarrow a}$ conserves state expectation values and is unitary (Messiah, 1966), and S usually taken as the identity, I. Then $U = B_{b\leftarrow a}$ and, of course, $B_{b\leftarrow a}$ is unitary, as it must be from Von Neumann's theorem and as shown for a general-type case in the first part of Section 2. Also, Wigner's theorem (Wick, 1965) guarantees that U is unitary if equation (3.3) holds.

However, for field theories exhibiting inequivalent representations of the CCR or CAR the typical result, as illustrated in Section 2, is

$$\langle \Phi_a | U | \Psi_a \rangle = \langle \Phi_a | S_{a\leftarrow b}B_{b\leftarrow a} | \Psi_a \rangle = 0 \tag{3.4}$$

Of course, the immediate question from this is whether equation (3.4) implies unitarily inequivalent representations or not, and we show here that it does indeed in the context of the physically relevant broken symmetry field theories. This has been alluded to in the literature, and we will use some of the developments of Guralnik, Kibble & Hagen (1968) to establish it.

In the context of the quantum field theory we are using, for U to be a unitary operator, three things are needed,

(1) $U(\alpha)U^\dagger(\alpha) = I$
(2) $U(\alpha_1)U(\alpha_2) = U(\alpha_1 + \alpha_2)$
(3) in our physical Hilbert space we assume that there is a translationally invariant, unique vacuum state (LSZ type of formalism), and U must do the right things on our states. Therefore, we must have

$$U_V(\alpha, t) | O \rangle = | O \rangle \tag{3.5}$$

and, in particular, for V becoming large (indicated by $V \to \infty$), we must have

$$\lim_{V \to \infty} (U_V(\alpha, t) \mid O\rangle) = 0 \qquad (3.6)$$

Or equivalently, if we consider any combination of operators A of the field theory being considered,

$$\lim_{V \to \infty} \langle O \mid U_V(\alpha, t) A U_V^\dagger(\alpha, t) \mid O\rangle = \langle O \mid A \mid O\rangle \qquad (3.7)$$

Here α represents a general constant parameterizing the transformation, V is the volume of the system, t is the time, and we can write U in general form as $U = \exp[i\alpha Q_V(t)]$, where $Q_V(t)$ is the generator of the transformation U in volume V.

Consider condition (3) above more closely and see what condition we must have for it to be satisfied. If we can show that

$$\frac{d}{d\alpha} [\langle O \mid U(\alpha, t) A U^\dagger (\alpha, t) \mid O\rangle] = 0 \text{ for all } \alpha \qquad (3.8)$$

then

$$\langle O \mid U(\alpha) A U^\dagger (\alpha) \mid O\rangle = \langle O \mid U(O) A U^\dagger (O) \mid O\rangle = \langle O \mid A \mid O\rangle \qquad (3.9)$$

which then gives us the equivalent statement of condition (3). Explicitly evaluating the derivative,

$$\frac{d}{d\alpha} [\langle O \mid U_V(\alpha, t) A U_V^\dagger(\alpha, t) \mid O\rangle] = i\langle O \mid U_V(\alpha, t) [Q_V(t), A] U_V^\dagger(\alpha, t) \mid O\rangle \qquad (3.10)$$

where $[Q_V(t), A]$ is the commutator of $Q_V(t)$ with A. The expression (3.10) will be zero if $[Q_V(t), A] = 0$. Then from equations (3.8) and (3.9) U is unitary. However, if $[Q_V(t), A] \neq 0$, then condition (3) is not satisfied, and U is *not* unitary. One will recognize this last condition as the one for $Q_V(t)$ to excite 'massless' modes, typically in the case where V becomes large (∞), and is a result of Goldstone's theorem (Lange, 1965). It is called the broken symmetry condition of the Goldstone theorem. Therefore, when the generator of a transformation is involved in a broken symmetry in quantum field theory, the corresponding U is *not* unitarily implementable in our physical space. In fact, Fabri & Picasso (1966) have shown that the generator Q does not exist even in the sense of a weak limit when condition (3) is violated.

Thus, we have from condition (3) that U is not a unitary operator when

$$U \mid O\rangle = \exp (i\alpha Q) \mid O\rangle \neq \mid O\rangle \qquad (3.11)$$

But this corresponds to $\langle O \mid U \mid O\rangle = 0$, since we then have the overlap of $\langle O \mid$ with a state of the space other than the unique $\mid O\rangle$. This is the result we were looking for.

In particular now, equation (3.4) must hold for $\mid \Phi_a\rangle = \mid \Psi_a\rangle = \mid O\rangle$, and thus, if we have equation (3.4) holding, from above we have inequivalent representations in our field theory. This is one of the links we wanted to establish.

Now further consider equation (3.4) to establish another link. Notice that the bodily transformation U in equation (2.19) is (1) an exponential and (2) antisymmetric in sign with respect to the adjoint operation. This is a typical result for many models and useful examples, and we will outline some others shortly. Due to these facts, there will certainly be nothing wrong with the bodily mapping in statements like equations (3.1) and (3.3). They will hold with no problems. (This is also guaranteed by writing the bodily transformation in the form $\exp[i\alpha Q]$, as done above.) Thus, the bodily mapping is all right.

Also, Wigner's theorem (Wick, 1965) says if we have equation (3.3) holding, then $S_{a\leftarrow b}B_{b\leftarrow a}$ is unitary. However, we have just shown above that it is not unitary in the field theories we are considering. Thus, we have reached a contradiction. Equation (3.1) is all right and equation (3.3) implies unitarity, but for the theories we are considering the corresponding operator is not unitarily implementable. Therefore, since equation (3.3) comes from equations (3.1) and (3.2), the breakdown in unitarity must come from equation (3.2), i.e., the subjective transformation is not unitarily implementable. Consequently, it can neither be the identity, I, which is generally always chosen in point quantum mechanics, nor any unitary operator, or else it would have a value in H_a. As a result, spaces 'a' and 'b' are using *mathematically different* Hilbert spaces, built upon different representations of the CCR or CAR.

A very good example to illustrate some of the above points is the fairly simple model Hamiltonian of the form

$$H = \hbar \sum_i \omega_i a_i^\dagger a_i + \hbar \sum_i \lambda_i (a_i^\dagger + a_i) \tag{3.12}$$

which is essentially the famous Van Hove model (Van Hove, 1952) (translated oscillators). For simplicity let's take the ω_i's equal, so that

$$H = \hbar\,\omega \sum_i a_i^\dagger a_i + \hbar \sum_i \lambda_i (a_i^\dagger + a_i) \tag{3.13}$$

If we define an operator

$$a = \sum_k (\lambda_k a_k)/\Lambda, \qquad \Lambda = [\sum_i (\lambda_i)^2]^{1/2} \tag{3.14}$$

and the corresponding adjoint, then

$$H = \hbar\,\omega a^\dagger a + \hbar\Lambda(a^\dagger + a) \tag{3.15}$$

The transformation which diagonalizes H is

$$U = \exp\left[\frac{\Lambda}{\omega}(a^\dagger - a)\right] \tag{3.16}$$

which is well known from the Van Hove model. In the limit that $\Lambda \to \infty$ or $\Sigma_i \lambda_i^2 \to \infty$, U is not unitarily implementable into the diagonalized space, and hence, equation (3.4) holds, i.e., zero overlap. This is another example of our above statement that zero overlap implies unitary inequivalence. Note again that the bodily transformation in equation (3.16) is an exponential and is antisymmetric in sign with respect to the adjoint operation, so equation (3.1) holds. Thus, again we know from our above link that two Hilbert spaces (the 'a' space and the diagonalized space) are incompatible with respect to the subjective transformation. They do not have mutually complete sets of states for description of the system.

A very practical example of these statements comes from a recent paper by Benson (1973) and one by Benson & Hatch (1973). Here a ferromagnetic representation is self-consistently selected for the Heisenberg-exchange Hamiltonian using Umezawa's methods of quantum field theory. Also, a paramagnetic representation is found, and the zero overlaps are specifically calculated, and it is discussed how these representations are inequivalent representations of the CAR. From one of our links above, the reason for this inequivalence is that one cannot subjectively transform the states of the ferromagnetic Hilbert space onto the states of the paramagnetic Hilbert space, since the set of paramagnetic states do not form a complete set of states for a ferromagnetic description. Thus, S is the ill-behaved part of the transformation.

4. *Conclusion*

Thus, from the above statements we have some more links in understanding the very important phenomena of inequivalent representations, and further insight into understanding some of the fundamental differences between point quantum mechanics and quantum field theory. In particular, although people frequently never worry at all about the structure of the subjective transformation, we have shown for quantum field theories that this consideration can be very important, and that one can expect inequivalent representations of the CCR or CAR when zero expectation values of the transformation operator occurs as in equation (3.4). Hopefully, further links will be forthcoming in the near future.

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